Algebraic Geometry Lecture 16 – Geometry of Surfaces

Mike Harvey

Recap (Divisors).

A *prime divisor* of a variety X is an irreducible, closed subvariety of codimension one. A *divisor* is a finite formal sum of prime divisors:

$$D = \sum_{\text{Prime divisors } P} n_P P, \qquad n_P \in \mathbb{Z}.$$

Let $f \in k(X)$ be a rational function, then

$$\mathrm{Div}(f) = \sum_{\text{Prime divisors } P} \mathrm{ord}_P(f) P$$

where

$$\mathrm{ord}_P(f) = \begin{cases} d \text{ if } f \text{ has a zero of order } d \text{ on } P, \\ -d \text{ if } f \text{ has a pole of order } d \text{ on } P, \\ 0 \text{ otherwise.} \end{cases}$$

A *principal divisor* is a divisor D such that $D = \mathrm{Div}(f)$ for some $f \in k(X)$.

A *canonical divisor* of X is

$$D = \mathrm{Div}(\omega) = \sum_{\text{Prime divisors } P} \mathrm{ord}_P(\omega) P$$

for a differential ω, where $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$ when $\omega = f dt_1 \wedge \ldots \wedge dt_n$.

Two divisors are called *linearly equivalent* if their difference is a principal divisor:

$$D \sim D' \qquad \Leftrightarrow \qquad D - D' = \mathrm{Div}(f) \text{ for some } f \in k(X).$$

We define

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)}$$

where

$$\mathrm{Div}(X) = \text{Group of divisors}$$
$$\mathrm{PDiv}(X) = \text{Group of principal divisors.}$$

Surfaces.

Proposition 1. *Let $X \subset \mathbb{P}^3$ be a surface, then any two plane sections (intersections of planes with X) are linearly equivalent, this gives the "hyperplane class" H in* $\mathrm{Pic}(X)$.

Proof. The hyperplane sections will be of the form

$$D_1 = X \cap \{\ell_1 = 0\}$$
$$D_2 = X \cap \{\ell_2 = 0\}$$

for linear equations ℓ_1, ℓ_2. Then

$$D_1 - D_2 = \mathrm{Div}\left(\frac{\ell_1}{\ell_2}\right).$$

□

Intersection Numbers.

Let D, D' be two prime divisors on a surface X that intersect transversely:

 rather than 

Then we define the intersection number of D and D' to be

$$D.D' := \#\{D \cap D'\}.$$

Properties

- Respects linear equivalence: if C, D, D' are divisors and $D \sim D'$ then
$$D.C = D'.C;$$
- Symmetric: if C, D are divisors then
$$D.C = C.D;$$
- Bilinear: If C, D_1, D_2 are divisors then
$$(D_1 + D_2).C = D_1.C + D_2.C.$$

"**Example**" We denote $D.D$ by D^2, but what is D^2? Find a divisor $D' \sim D$ such that $D' \neq D$, then we define $D^2 = D'.D$.

Example Let $X = \mathbb{P}^2$, so $\mathrm{Pic}(X) \cong \mathbb{Z}$. Let h be a generator of $\mathrm{Pic}(X)$, so h is a "line-class". Any two lines are linearly equivalent and two distinct lines meet in one point, so $h^2 = 1$. Now let C, D be curves of degree m, n respectively. So

$$C \sim mh \qquad D \sim nh.$$

We then have

$$\begin{aligned} C.D &= mh.nh \\ &= (mn)h^2 \\ &= mn. \end{aligned}$$

Example Let X be a non-singular quadric surface in $\mathbb{P}^3$, then $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let ℓ be a generating line corresponding to $(1,0)$ and m be a generating line corresponding to $(0,1)$. Then $\ell^2 = m^2 = 0$ and $\ell.m = 1$ since any two lines in the same family are skew and two lines from opposite families meet in a point[1]. Now let C be a curve of type (a,b) and C' be a curve of type (a',b'). So

$$C \sim a\ell + bm \qquad C' \sim a'\ell + b'm.$$

Then

$$\begin{aligned} C.C' &= (a\ell + bm).(a'\ell + b'm) \\ &= aa'\ell^2 + bb'm^2 + (ab' + ba')\ell.m \\ &= ab' + ba'. \end{aligned}$$

Adjunction Formula.

Set:

$$\begin{aligned} C &\text{ - a smooth curve on a surface } X \\ K &\text{ - canonical divisor of } X \\ g &\text{ - genus of } C, \end{aligned}$$

then the adjunction formula is

$$2g - 2 = C.(C + K).$$

Example Let C be a degree d curve in $\mathbb{P}^2$. Exercise: Show that $\deg K_{\mathbb{P}^2} = -3$. So $\deg C = d$, $\deg(C + K) = d - 3$, and by the adjunction formula

$$2g - 2 = d(d - 3)$$

so the genus of C is

$$g = \tfrac{1}{2}(d - 1)(d - 2).$$

[1]See Hartshorne p.361.